Rolling Sphere Problems and Elastic Curves:
Completely Integrable Hamiltonian Systems
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1 Abstract.

In this lecture I will define a class of variational problems, called the rolling sphere problems **(RSP)** on spaces of constant curvature. These problems can be reformulated as optimal control problems on Lie groups $G_\epsilon, \epsilon = 0, 1, -1$, with $G_0 = \mathbb{R}^n \times SO_{n+1}$ and $\mathbb{R}^n \times SO(1,n)$ for the Euclidean cases, $G_1 = SO_{n+1} \times SO_{n+1}$ in the spherical case and $G_{-1} = SO_0(1,n) \times SO_0(1,n)$ in the hyperbolic case.

The Maximum Principle then singles out the appropriate Hamiltonian vector fields on the cotangent bundle T^*G, realized as the product $G \times \mathfrak{g}^*$ with $\mathfrak{g}^*$ equal to the dual of the Lie algebra $\mathfrak{g}$ of G, whose integral curves (extremals) project onto the optimal trajectories (minimizers). We show that each minimizer is the projection of a normal extremal curve, and hence the study reduces to the analysis of a single Hamiltonian vector field $\vec{\mathcal{H}}$ on T^*G.

Remarkably, the Hamiltonian equations associated with each of the above mentioned rolling sphere problems are governed by a single set of equations on $\mathfrak{g}_\varepsilon, \varepsilon = \pm 1$ with $\mathfrak{g}_1 = so_{n+1}(R)$ and $\mathfrak{g}_{-1} = so(1,n)$, expressed through the Cartan decomposition $\mathfrak{g}_\varepsilon = \mathfrak{p}_\varepsilon \oplus \mathfrak{k}$ as follows:

$$\frac{dA}{dt} = 0 \qquad \frac{dK}{dt} = [U_\varepsilon, P] \qquad \frac{dP}{dt} = [U_\varepsilon, K] \qquad U_\varepsilon = A + P, \tag{1}$$

where A and P belong to Cartan space $\mathfrak{p}_\varepsilon$ and K belongs to the subalgebra $\mathfrak{k}$.

The above equations admit a Laxian representation

$$\frac{dL_\lambda}{dt} = \left[\lambda^{-1}(A + P), L_\lambda\right] \tag{2}$$

with $L_\lambda = A + P + \lambda K - \lambda^2 A$ that generate constants of motion

$$\phi_{\lambda,k} = tr\left(\left(P + \lambda K + (1 - \lambda^2)A\right)^k\right). \tag{3}$$

This observation makes contact with the theory of integrable Hamiltonian systems from which it follows that (**RSP**) are completely integrable. Among these integrals of motion

$$I_4 = \|K\|^2\|A + P\|^2 - \|\,[A + P, K]\,\|^2 \tag{4}$$

is particularly significant in the sense that the projections of the extremal curves of **RSP** on the stationary manifold M_ε that conform to $I_4 = 0$ coincide with the elastic curves.

The solutions make connections with mechanical tops, planar pendulum for $n = 2$, and spherical pendulum for $n = 3$.